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Sensitivity of the Transmission Zeros of Flexible Space Structures

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The uncertainties inherent in the dynamics of flexible spacecraft make sensitivity and robustness questions very important when designing vibration isolation systems for these vehicles. One technique recently proposed for this problem is pole/zero cancellation, in which state feedback is used to make as many closed-loop modes as possible unobservable at the points on the structure where vibrations are to be minimized. Study of the robustness properties of this approach first requires an analysis of the sensitivity of the transmission zeros of second-order structural models. These questions are addressed in this paper, where it is proved (first in terms of partial derivatives and then condition numbers) that the sensitivities of the transmission zeros of any structure with collocated sensors and actuators are closely related to those of the poles of the system. Furthermore, the closed-loop poles produced by applying pole/zero cancellation to such a structure are shown to have sensitivities approaching those of the zeros, so these too are approximately given from the sensitivities of the open-loop poles. Finally, these points are illustrated by simple examples.

Introduction

THE uncertain nature of the dynamics of flexible space structures (FSS) makes sensitivity and robustness questions of great importance for such systems. This results from the fact that preflight tests of an FSS must be carried out with the structure supported against the effects of gravity. The measured repsonse is therefore not that of the structure itself but rather of the structure plus support system. Furthermore, air resistance introduces a damping mechanism not present in space. The actual on-orbit dynamics of an FSS may therefore turn out to be significantly different from the model derived before launch. Any control system designed for such a spacecraft must therefore be robust enough to deal with appreciable perturbations to the open-loop system.

One technique that has recently been proposed[1] for FSS vibration suppression is that of pole/zero cancellation. This uses state feedback to make as many closed-loop modes as possible unobservable at the points on the structure where vibrations are to be minimized. The dynamic response at these points is therefore due solely to the remaining uncanceled modes; these can be made to decay as quickly as desired by application of dynamic output feedback. This approach seems quite promising for certain vibration isolation problems, but a practical question not addressed in Ref. 1 is that of robustness. This is the topic of the present paper.

Central to this study is an analysis of the sensitivity of the transmission zeros[2] of second-order structural models. The zeros of such systems have recently been shown[3,4] to have special properties that do not apply for general linear systems. This strongly suggests that the sensitivities of these zeros will also exhibit interesting properties, and yet this question does not appear to have been studied to date. The starting point taken in the analysis presented here is the classical result[5] that the zeros of any undamped structure with a single compatible (physically collocated and coaxial) sensor/actuator pair alternate with its poles along each half of the imaginary axis. Since this must also hold for any perturbed system, it suggests that the zeros can be no more sensitive to perturbations than are the poles. Such a result can in fact be proved using partial derivatives and is given in this paper. To be precise, it is shown that perturbing a system pole by some small amount shifts all zeros by amounts of smaller magnitude and the same sign. Furthermore, for the special case of a structure with widely spaced natural frequencies, the perturbation of any one zero is approximately only dependent on those of the two poles it lies between, exactly as should be expected from the preceding argument.

It was shown in Ref. 3 that the classical pole/zero interlacing result can be generalized to modally damped flexible structures with an arbitrary number of compatible sensor/actuator pairs. The poles of the structure now define a region in the left half of the complex plane in which all transmission zeros must lie (see Fig. 1), regardless of where the sensor/actuator pairs are actually positioned. This suggests that some relationship between the sensitivities of the zeros and those of the poles should also exist in this case. It is proved in this paper that such

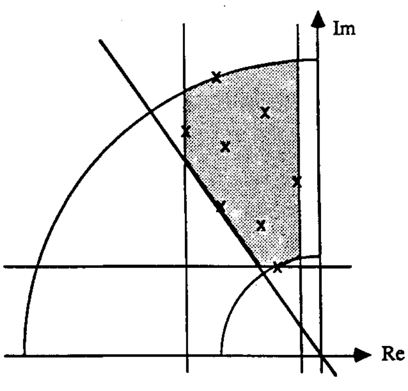

Fig. 1 Zero region for modally damped structure.

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results do indeed apply: the undamped single-input/single-output (SISO) partial derivative relation generalizes partially to this case, whereas the eigenstructure condition number technique of Ref. 6 yields more complete conclusions. The latter approach also leads to a proof of the fact that the closed-loop poles produced by applying pole/zero cancellation to such a structure will have sensitivities approaching those of the transmission zeros. These poles will therefore typically be no more sensitive to perturbations than are the poles of the open-loop system, a very satisfactory result. These points are illustrated by application to simple SISO and multi-input/multi-output (MIMO) examples.

FSS Transmission Zero Properties

Consider an n-mode model for the structural dynamics of a nongyroscopic, noncirculatory FSS with m compatible sensor/actuator pairs. (By compatible we mean that the direction of the linear/angular motion measured by each sensor is the same as that of the force/torque applied by the actuator that is collocated with it.) This model can be written in modal form[7] as

$$\ddot{\eta} + C\dot{\eta} + \mathrm{diag}(\omega_i^2)\eta = \Phi^T V u$$

$$y = W_r\Phi\dot{\eta} + W_d\Phi\eta \tag{1}$$

where η is the vector of modal coordinates, u that of applied actuator inputs, and y that of sensor outputs. $\Phi=(\phi_{ij})$ is the $(m\times n)$ modal influence matrix (ϕ_{ij} is the value of mass-normalized mode j, corresponding to natural frequency ω_j, at sensor/actuator station i), $C=C^T\geq 0$ is the damping matrix in modal form, and V is an $(m\times m)$ nonsingular matrix describing how actuator inputs translate to the physical forces applied to the structure. Typically, V is diagonal if the forces at each station are independent, while it has a column of the form $(\mathbf{0},a,\mathbf{0},-a,\mathbf{0})^T$ if equal and opposite reaction forces are applied between two stations. The matrices W_r and W_d are defined similarly for rate and displacement sensors, respectively.

Taking Laplace transforms, we obtain the polynomial matrix representation[8]

$$P(s)\eta(s) = \Phi^T V u(s)$$

$$y(s) = W(s)\Phi\eta(s) \tag{2}$$

for the given FSS, where $P(s)=s^2I+sC+\mathrm{diag}(\omega_i^2)$ and $W(s)=sW_r+W_d$. Note that $P(s)$ is symmetric; i.e., Eq. (2) respects the special structure of the LSS equations of motion. This is in contrast to the state-space representation $\{A,B,C\}$ obtained by setting $x=(\dot{\eta}^T,\eta^T)^T$, where A no longer preserves this valuable symmetric structure.

As long as its actuators and sensors have been positioned in such a way as to make it completely controllable and observable, i.e., so that each mode can be both excited and sensed, the transmission zeros of this invertible system are[2,9] those s_i that reduce the rank of the system matrix:

$$S(s) = \begin{pmatrix} P(s) & \Phi^T V \\ -W(s)\Phi & 0 \end{pmatrix} \tag{3}$$

Associated with each transmission zero s_i is a zero mode shape η_i that satisfies

$$S(s_i)\begin{pmatrix} \eta_i \\ -u_i \end{pmatrix} = \mathbf{0} \tag{4}$$

where η_i can be regarded as the solution of a constrained modes problem,[10] with the constraint being that the mode has zero deflection/slope at each linear/angular sensor location.

A simple dynamical interpretation of transmission zeros should help to avoid an element of confusion often present in discussions of the zeros of multi-input systems. A comparison of Eqs. (2-4) shows that if s_i is a transmission zero, then applying the nonzero input $\exp(s_i t)u_i$ to the system with modal initial condition η_i gives rise to the identically zero output $y(s_i)=W(s_i)\Phi\eta_i$. This property is the dual of that of the system poles (or resonances), which are frequencies at which it is possible to obtain a nonzero output evolving exponentially with time in response to an identically zero input, again for suitable initial conditions. It is important to note that the quantities sometimes referred to erroneously as the transmission zeros of a multi-input system, namely, those frequencies that make zero the scalar transfer functions between individual inputs and outputs, do *not* in general have any such physical interpretation.[9,11]

It can be shown from Eq. (3) that $\det[W(s)]=0$ specifies q_s finite sensor zeros ($0\leq q_s\leq m$), there are $2m-q_s$ zeros at infinity, and the remaining $2(n-m)$ structural zeros are defined by the physical properties of the structure and the positions chosen for sensor/actuator pairs. The structural zeros always lie in the left half-plane (LHP); furthermore, if as is often the case the structure is modally damped[7] with damping ratios $\{\zeta_i\}$, i.e., $C=\mathrm{diag}(2\zeta_i\omega_i)$, then[3] the poles $-\zeta_i\omega_i\pm j\omega_i\sqrt{[1-\zeta_i^2]}$ of the system define a portion of the LHP in which all these zeros must lie, regardless of the specific locations chosen for sensor/actuator pairs. This generic result, consisting of upper and/or lower bounds on the real and imaginary parts, moduli, and damping ratios of all zeros, is a consequence of the special form of the equations of motion of structural dynamics; it can be regarded as a generalization of the classical observation[5] that the zeros of a single-input/single-output undamped structure alternate with its poles along the imaginary axis. It admits a very simple graphical interpretation, as shown in Fig. 1 for an arbitrary distribution of poles x.

FSS Transmission Zero Derivatives

We now turn to the main problem to be studied in this paper, that of characterizing the sensitivity of the transmission zeros of a flexible structure to system perturbations. As a first approach, the partial derivatives of the zeros with respect to the modal parameters of the structure are derived in this section. Of course, the actual physical variations in a structure will be in its mass and stiffness quantities; these will in turn affect its modal parameters. The sensitivities of the zeros to physical variations can thus be conveniently, if somewhat indirectly, represented in terms of their sensitivities to modal variations. The zeros derivatives provide a very complete sensitivity description; however, their number may become excessive for large system models. (This point will be returned to in the next section.)

For clarity, the first case to be considered is the particularly simple one of an undamped single-input/single-output structure. As noted at the end of the last section, the structural zeros of such a system interlace its poles along the imaginary axis. Furthermore, this property must still hold after any parameters of the system are perturbed. Therefore, the changes in the zeros induced by such a perturbation are presumably of comparable size to, and the same sign as, the changes in the poles. This result will now be proved formally.

For a single-input/single-output undamped system with zero $s_i=jz_i$, Eq. (4) becomes

$$\begin{bmatrix} \mathrm{diag}(\omega_i^2-z_i^2) & \phi \\ -\phi^T & 0 \end{bmatrix}\begin{pmatrix} \eta_i \\ -u_i \end{pmatrix} = \mathbf{0} \tag{5}$$

where we have taken $V=W(s)=1$ without loss of generality and written the $(n\times 1)\Phi^T$ in standard column vector form as ϕ. Considering the first row block of this equation gives

$$\eta_i = \mathrm{diag}(\omega_j^2-z_i^2)^{-1}\phi u_i \tag{6}$$

Substituting this into the last row then yields $\phi^T \operatorname{diag}(\omega_j^2 - z_i^2)^{-1}\phi u_i = 0$ or, writing $\phi = (\phi_1, \ldots, \phi_n)^T$ and noting that the scalar u_i is nonzero,

$$\sum_{j=1}^{n} \left[\phi_j^2 / (\omega_j^2 - z_i^2)\right] = 0 \tag{7}$$

As this expression is identically zero, its partial derivative with respect to ω_k^2 must also be zero for any k. Carrying out this differentiation term by term and rearranging can be shown to yield the relation

$$\partial(z_i^2)/\partial(\omega_k^2) = \left[\phi_k^2/(\omega_k^2 - z_i^2)^2\right] \Big/ \sum_{j=1}^{n} \left[\phi_j^2/(\omega_j^2 - z_i^2)^2\right] \tag{8}$$

[The left-hand side (LHS) has been left in terms of z_i^2 and ω_k^2 for simplicity, but $\partial z_i / \partial \omega_k$ is of course easily obtained from it, if desired, by multiplying by ω_k / z_i.]

The important point to note about Eq. (8) is that all of the terms in brackets are guaranteed non-negative. It therefore follows that $\partial(z_i^2)/\partial(\omega_k^2)$ must also be non-negative; in fact, it must lie in the interval [0,1]. Thus, if ω_k^2 increases by some small amount ϵ, each z_i^2 must also increase, with the amount of increase not being greater than ϵ. Furthermore, the sum of the increases in all z_i^2 is, to a first approximation, just ϵ; this follows from the fact that

$$\sum_{i=1}^{n} \partial(z_i^2)/\partial(\omega_k^2) = 1$$

A final consequence of Eq. (8) is that $\partial(z_i^2)/\partial(\omega_k^2)$ increases as z_i approaches ω_k, i.e., as the separation between the pole and zero of interest decreases. These results are all in agreement with the pole/zero interlacing property of undamped SISO structures. In particular, the final point reflects the fact that, to preserve interlacing under perturbations, a zero must be more sensitive to changes in the poles it lies between than to changes in poles it is far away from.

Similar conclusions can also be drawn concerning the sensitivities of the zeros to changes in the modal influence vector ϕ. Differentiating Eq. (7) with respect to ϕ_k^2, noting that this derivative must be zero, and rearranging gives

$$\partial(z_i^2)/\partial(\phi_k^2) = 1 \Bigg/ \left\{ (z_i^2 - \omega_k^2) \sum_{j=1}^{n} \left[\phi_j^2/(\omega_j^2 - z_i^2)^2\right] \right\} \tag{9}$$

Just as was the case for Eq. (8), this expression increases as z_i approaches ω_k. A zero is therefore more sensitive to changes in the modal influence coefficient corresponding to a nearby pole than to changes in that corresponding to a far pole, as is to be expected. On the other hand, Eq. (9) differs from Eq. (8) in that its sign is not always positive; rather, it depends on whether z_i is greater than or less than ω_k.

Another question of interest is that of characterizing the sensitivities of the zero mode shapes of the system; this is analogous to studying the derivatives, not just of eigenvalues, but also of the associated eigenvectors. Closed-form results can again be obtained for this problem, based on the expression given by Eq. (6) for the ith zero mode shape. Differentiating this with respect to each ω_k and ϕ_k yields

$$\partial\eta_i/\partial\omega_k = -\boldsymbol{e}_k \cdot 2u_i\omega_k\phi_k/(\omega_k^2 - z_i^2)^2 \tag{10}$$

and

$$\partial\eta_i/\partial\phi_k = \boldsymbol{e}_k \cdot u_i/(\omega_k^2 - z_i^2) \tag{11}$$

where $\boldsymbol{e}_k$ is the kth column of the identity matrix. It can be seen that the separation between the ith zero and kth pole affects these expressions in precisely the same way as it did the zero derivatives. Also, changes in ω_k and ϕ_k of course only influence the kth entry of η_i, i.e., the contribution of mode k to the zero mode of interest.

As a final point on the undamped SISO case, note that Eqs. (8) and (9), taken over all i and k, each define $n(n-1)$ derivatives; Eq. (10) provides another $2n(n-1)$. This number of parameters may well be prohibitively high when n is large, as is typical of FSS. A derivative-based approach is therefore probably most suitable for studying the sensitivity of certain specified zeros in detail, rather than for obtaining an idea of the sensitivity of the overall zero structure of the system; the condition number method of the next section is more suited to the latter problem.

Many of the single-input sensitivity results derived previously generalize quite directly to the multi-input/multi-output case. This can be shown by considering the ith transmission zero $s_i = z_i \exp(j\theta_i) = x_i + jy_i$ of a modally damped MIMO structure. The proof given in Ref. 3 of the relationships between the absolute values and real parts of poles and zeros (shown in Fig. 1) was based on the weighted-mean equations

$$z_i^2 = \sum_{j=1}^{n} \gamma_j \omega_j^2 \tag{12}$$

and

$$x_i = \sum_{j=1}^{n} \gamma_j(-\zeta_j\omega_j) \tag{13}$$

where the $\{\gamma_j\}$ are non-negative parameters satisfying

$$\sum_{j=1}^{n} \gamma_j = 1$$

Differentiating Eq. (12) with respect to ω_k^2 then clearly gives

$$\partial(z_i^2)/\partial(\omega_k^2) = \gamma_k \tag{14}$$

where, just as in Eq. (8), this is a quantity in the interval [0,1]. In a similar fashion, Eq. (13) implies that a small change ϵ in the real part $-\zeta_k\omega_k$ of the kth pole will lead to a change of the same sign, but lesser magnitude, in the real part of each zero, with the summed change over all zeros being approximately ϵ. Thus, the absolute values and real parts of the transmission zeros of a damped MIMO structure have sensitivity properties with respect to the poles that are entirely analogous to those of the zeros of an undamped SISO one. Combining these results, it is clearly possible to determine the changes to the real and imaginary parts of any zero that will occur if the poles are altered by known small (complex) amounts.

On the other hand, tractable explicit expressions cannot now be obtained for the zero derivatives with respect to the entries of the modal influence matrix Φ. This is a consequence of the fact that, while the parameters $\{\gamma_j\}$ are independent of $\{\omega_j\}$ and $\{\zeta_j\}$, they do depend indirectly on Φ. Likewise, closed-form expressions for the derivatives of the zero mode shapes of a structure do not apear to be tractable in the MIMO case.

It is possible, however, to compute the numerical values of all these derivatives by means of the eigenstructure sensitivity techniques of Ref. 12. That paper was motivated by the desire to be able to analyze the effects on the response of a truss structure of changes in its joint stiffness distribution, and contained extensive work on the derivatives of eigevalues (possibly repeated) and their associated eigenvectors. Applying this technique to the current problem allows values to be computed for the derivatives of the zeros and zero mode shapes of a MIMO structure with respect not only to its modal parameters but also to any desired physical quantities (e.g., stiffness properties). The interested reader is referred to Ref. 12 for details; all that will be derived here is the necessary initial reformulation of Eq. (3) as a standard eigenvalue problem. This makes use of the canonical form introduced in Ref. 4 based on the QR decomposition[13] of the full column rank Φ^T, i.e., $\Phi^T = QR$ with Q orthogonal and $R = (R_1^T, 0)^T$ upper triangular and full rank.

Substituting this factorization into Eq. (3) gives

$$S(s) = Q_s \begin{bmatrix} Q^T P(s) Q & RV \\ -W(s)R^T & 0 \end{bmatrix} Q_s^T \tag{15}$$

where $Q_s = \mathrm{diag}(Q, I)$ is orthogonal. Thus, partitioning Q as (Q_1, Q_2), Q_1 $(n \times m)$, we have

$$S(s) = Q_s \begin{bmatrix} Q_1^T P(s) Q_1 & Q_1^T P(s) Q_2 & R_1 V \\ Q_2^T P(s) Q_1 & Q_2^T P(s) Q_2 & 0 \\ -W(s) R_1^T & 0 & 0 \end{bmatrix} Q_s^T \tag{16}$$

so the structural zeros are clearly simply those s_i that make singular the $(n-m)\times(n-m)$

$$Q_2^T P(s) Q_2 = s^2 I + sC_2 + K_2 \tag{17}$$

where $C_2 = Q_2^T C Q_2$ and $K_2 = Q_2^T \mathrm{diag}(\omega_i^2) Q_2$. Note that Q_2 depends in an explicit way on the positions selected for sensor/actuator pairs; its columns form an orthonormal basis for the orthogonal complement[13] of Φ^T, or equivalently $Ker(\Phi)$.

This approach to calculating the structural zeros clearly gives them as the $2(n-m)$ solutions of the standard eigenvalue problem $Ax = sx$, where

$$A = \begin{pmatrix} -C_2 & -K_2 \\ I & 0 \end{pmatrix} \tag{18}$$

Furthermore, the eigenvectors of A can be shown to completely specify the zero mode shapes of the system, assuming the structural and sensor zeros are mutually distinct. This mild restriction is clearly always satisfied for the typical case of $W(s) = sW_r + W_d$ diagonal with W_r and W_d non-negative, as all sensor zeros are then nonpositive real. As a final point, note that Eq. (18) was used in Ref. 4 as the basis for an algorithm to compute the transmission zeros of an FSS that is at least 60 times as fast as the general-purpose zeros method of Ref. 14 when applied to an undamped structure and 15 times as fast for a lightly damped one.

FSS Transmission Zero Condition Numbers

We shall now study the sensitivity of the transmission zeros of flexible structures by analyzing their condition numbers. The condition number of a numerical quantity is[15] a real number that provides an upper limit on the amplification between a perturbation in the input data and the resulting perturbation in the quantity of interest. For instance, the sensitivity of the inverse of a matrix A to changes in A is[16] measured by $\kappa(A) = \|A\| \cdot \|A^{-1}\|$, the condition number with respect to inversion of A. It should be noted that the condition number of a quantity provides less detailed sensitivity information than do its partial derivatives. There are two reasons for this: the condition number measures the magnitude of change but not its direction, and changes in all input data parameters are lumped together rather than considered separately. For instance, $\kappa(A)$ does not specify which of the elements of A have been perturbed. It can also be seen by considering $\kappa(A)$ that different choices for the matrix norm used to measure changes in A and A^{-1} result in different values for this condition number. Thus, condition numbers must be regarded as *guides* to sensitivity behavior; their precise values are not terribly important. However, despite these seeming limitations, condition numbers have been found[6] to provide a good compromise between the conflicting requirements of clarity and completeness in the sensitivity analysis of many problems.

In the present application, we shall make use of the fact that Eq. (18) gives the transmission zeros and zero mode shapes of an FSS as the solutions of a standard eigen-value problem. The well-known condition number theory developed for the eigenstructure problem by Wilkinson[6] can therefore be applied directly to the zeros problem. In particular, if the matrix A has a distinct eigenvalue λ_i with associated right eigenvector x_i $(Ax_i = \lambda_i x_i)$ and left eigenvector y_i $(y_i^H A = \lambda_i y_i^H$; H denotes conjugate transpose), then the condition number of λ_i is

$$c_i = \|x_i\|_2 \cdot \|y_i\|_2 / |y_i^H x_i| \tag{19}$$

(Note that c_i is just $|\sec\theta|$, where θ is the angle between x_i and y_i.) Furthermore, the condition number of the corresponding eigenvectors is[16] δ_i^{-1}, where

$$\delta_i \leq \min\{|\lambda_i - \lambda_j| : j \neq i\} \tag{20}$$

Thus, it can be seen that an eigenvalue will be ill conditioned (sensitive to perturbations in A) if the corresponding left and right eigenvectors are nearly orthogonal, giving a small value for the quantity $|y_i^H x_i|$. The eigenvectors associated with λ_i will be ill conditioned if another eigenvalue lies near it, giving a small value for δ_i. It should be noted though that, as Eq. (20) is merely an inequality, it is possible to have well-separated eigenvalues and yet ill-conditioned eigenvectors; see Refs. 6 and 17 for further details and a striking example.

To be able to use Eq. (19) to compute the condition numbers of the zeros of an FSS, we must first find the left and right eigenvectors of a matrix of the form given by Eq. (18). The expressions that will be obtained can clearly also be used to determine the conditioning of the poles of the structure, as Eq. (1) can be rewritten in the form of Eq. (18) with C_2 replaced by C and K_2 by $\mathrm{diag}(\omega_i^2)$. In fact, one result of the following analysis will be to tie together the conditioning of the poles and zeros of a lightly damped structure in a very simple and direct way.

Consider the relation $Ax_i = \lambda_i x_i$ first, with A given by Eq. (18) and x_i written as $(x_a^H, x_b^H)^H$. The second row block of this equation gives $x_a = \lambda_i x_b$, which when substituted into the first row block yields

$$[\lambda_i^2 I + \lambda_i C_2 + K_2] x_b = 0 \tag{21}$$

This determines x_b, and so x_i, up to a scalar multiplier; without loss of generality we may take $\|x_b\|_2 = 1$, giving

$$\|x_i\|_2 = \sqrt{[1 + |\lambda_i|^2]} \tag{22}$$

Writing $y_i^H = (y_a^H, y_b^H)$, we can carry out a similar process on the equation $y_i^H A = \lambda_i y_i^H$. Expanding this we have $(-y_a^H C_2 + y_b^H, -y_a^H K_2) = \lambda_i (y_a^H, y_b^H)$, so multiplying the first column block by λ_i and substituting for $\lambda_i y_b^H$ from the second column block yields

$$y_a^H [\lambda_i^2 I + \lambda_i C_2 + K_2] = 0 \tag{23}$$

Note that C_2 and K_2 are symmetric; thus, comparison of Eqs. (21) and (23) shows that we can take $y_a^H = x_b^T$. Also, $\lambda_i y_b^H = -y_a^H K_2 = y_a^H (\lambda_i^2 I + \lambda_i C_2)$ by Eq. (23), so $y_b^H = x_b^T (\lambda_i I + C_2)$ and

$$\|y_i\|_2 = \sqrt{[1 + \|x_b^T (\lambda_i I + C_2)\|_2^2]} \tag{24}$$

The only quantity still needed before we can evaluate the condition number c_i is the inner product $y_i^H x_i$. This can now be given in terms of the known subvector x_b as follows:

$$\begin{aligned} y_i^H x_i &= y_a^H x_a + y_b^H x_b \\ &= x_b^T (\lambda_i x_b) + [x_b^T (\lambda_i I + C_2)] x_b \\ &= 2\lambda_i (x_b^T x_b) + x_b^T C_2 x_b \end{aligned} \tag{25}$$

Equations (21), (22), (24), and (25) provide all of the data required to compute the condition numbers of the zeros (or poles, for suitable modified A) of any flexible structure. They also lead to even simpler results for the special case of an undamped structure. In fact, the numerical results of the general case are then replaced by very straightforward closed-form expressions that also apply approximately (with relative error of order $\{\zeta_i^2\}$) for the typical FSS case of a lightly damped structure. Setting $C_2=0$ does not alter Eq. (22), but it reduces Eq. (24) to $\|y_i\|_2=\sqrt{[1+|\lambda_i|^2]}=\|x_i\|_2$ and Eq. (25) to $y_i^H x_i=2\lambda_i$, where we have made use of the fact that x_b is real (and so $x_b^T x_b=\|x_b\|_2^2=1$) for $C_2=0$. Equation (19) then becomes just

$$c_i=(1+|\lambda_i|^2)/(2|\lambda_i|) \tag{26}$$

A graph of this function is given in Fig. 2. As noted previously, this expression holds not only for the zeros ($\lambda_i=jz_i$) but also for the poles ($\lambda_i=j\omega_i$) of any undamped flexible structure. It is therefore encouraging to observe that Fig. 2 is in general agreement with the conditioning results obtained in Ref. 18 concerning the poles of undamped structures. An heuristic argument was developed there, based on application of Stewart's condition number work for the generalized eigenvalue problem[19] to assumed-modes models, which suggested that condition numbers should, by and large, increase for increasing natural frequency.

The most important conclusion to be drawn from Eq. (26) is that the condition numbers of the zeros of an undamped structure are closely related to those of its poles. The zeros of such a structure with m inputs satisfy bounds of the form[3] $\omega_i \le z_i \le \omega_{i+m}$, so Fig. 2 shows that the condition number of z_i is (apart from a minor complication in the vicinity of $|\lambda_i|=1$) bounded above and below by those of ω_i and ω_{i+m}. Furthermore, this conclusion also holds, approximately, for the typical FSS case of light damping; Eq. (21) implies that x_b is then predominantly real, and so Eqs. (24) and (25) only undergo small changes. The transmission zeros of any FSS with compatible sensor/actuator pairs are therefore only about as sensitive to system perturbations as are its poles, a result of great practical importance.

Conditioning of Pole/Zero Cancellation

The results of the preceding section will now be applied to the study of the robustness properties of a vibration suppression technique recently derived[1] for FSS. This pole/zero cancellation approach is based on using state feedback to make as many closed-loop modes as possible unobservable, so that they do not appear in the sensed outputs. A practical reason for wishing to do this arises[20] if the open-loop system has a slowly decaying mode that prevents fast output regulation; it is likely to require less control effort to make this undesirable mode unobservable than it would to speed up its response substantially. In the frequency domain, each such mode can be shown to correspond to a closed-loop pole and associated mode shape that have been made equal to some transmission zero and its associated zero mode shape. This close connection between the eigenstructure of the closed-loop system and the zero structure of the open-loop one makes it likely that conclusions can be drawn, based on the zeros sensitivity results of the last section,

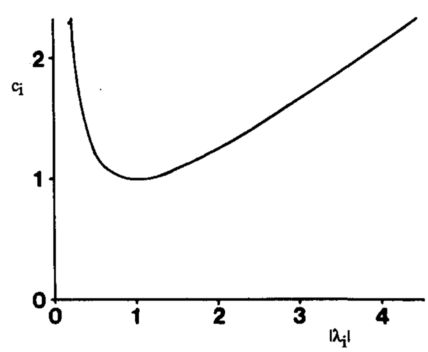

Fig. 2 Pole and zero condition numbers.

concerning the robustness of pole/zero cancellations. We shall now prove that this is indeed the case.

In FSS terms, linear state feedback[2] becomes $u=F_r\dot\eta+F_d\eta+Gv$, a combination of rate and displacement feedback together with an external input v. This can be written in the frequency domain as

$$u(s)=F(s)\eta(s)+Gv(s) \tag{27}$$

where $F(s)=sF_r+F_d$. (Of course, in practical applications the entire state is not directly known, and so it must be reconstructed by means of an observer[2].) Applying this control to the system described by Eq. (2) clearly produces a closed-loop system with polynomial matrix representation:

$$[P(s)-\Phi^T VF(s)]\eta(s)=\Phi^T VGv(s)$$

$$y(s)=W(s)\Phi\eta(s) \tag{28}$$

Now it is well known[2] that, if the original system is completely controllable, all closed-loop poles can be arbitrarily assigned by suitable choice of $F(s)$; the transmission zeros, on the other hand, are invariant under state feedback. Furthermore, while controllability cannot be altered by state feedback, observability, i.e., $F(s)$, can be chosen so as to make certain closed-loop modes unobservable. (The simpler output feedback control law, by contrast, cannot alter controllability or observability.) The mechanism by which state feedback creates unobservable modes can best be described as follows. Equation (28) shows that any closed-loop pole s_i and its corresponding mode shape η_i must satisfy $[P(s_i)-\Phi^T VF(s_i)]\eta_i=\mathbf{0}$; if this mode is to be unobservable it must also satisfy $y(s_i)=W(s_i)\Phi\eta_i=\mathbf{0}$. But comparison with Eqs. (3) and (4) shows that these two conditions are precisely the transmission zero defining relations [for $v=\mathbf{0}$, i.e., state feedback $u_i=F(s_i)\eta_i$], and so s_i and η_i must in fact be a transmission zero and its corresponding zero mode. This observation gives rise to the frequency domain notion that unobservable modes correspond to pole/zero cancellations in the closed-loop system; state feedback is considered to be chosen so as to shift an open-loop pole to the position of a transmission zero, cancelling with it in the closed-loop transfer matrix. This representation has the advantage that it specifies directly, from the zeros, the number of modes that can be made unobservable as well as the values of the corresponding closed-loop poles. It is interesting to note that the zero bounds illustrated by Fig. 1 guarantee that no structural zero will lie far from the open-loop poles, so the feedback gain required to shift poles to all the zero locations is never likely to be prohibitively large.

In the transformed coordinate system of Eq. (16), the closed-loop denominator matrix of Eq. (28) becomes

$$P_F(s)=\begin{bmatrix} [Q_1^T P(s)Q_1-R_1 VF_1(s)] & [Q_1^T P(s)Q_2-R_1 VF_2(s)] \\ Q_2^T P(s)Q_1 & Q_2^T P(s)Q_2 \end{bmatrix} \tag{29}$$

where $F_1(s)=F(s)Q_1$ and $F_2(s)=F(s)Q_2$. Now it has been shown[1] that, for full pole/zero cancellation, $F_2(s)$ must be chosen so that the top right block of $P_F(s)$ is zero; since $R_1 V$ is square and of full rank, such a choice always exists and is unique. By contrast, $F_1(s)$ is arbitrary; it is equivalent to dynamic output feedback and can be used to freely place those poles that remain observable in the closed-loop system. It will be shown next that the locations chosen for these residual poles play a central role in determining the robustness of pole/zero cancellation.

To apply the condition number approach of the last section to the unobservable closed-loop poles of Eq. (29) [with $F_2(s)$

as just described], $P_F(s)$ must first be rewritten in first-order form as [c.f., Eq. (18)]

$$A_F = \begin{bmatrix} -C_1 & 0 & -K_1 & 0 \\ -C_3 & -C_2 & -K_3 & -K_2 \\ & I & & 0 \end{bmatrix} \tag{30}$$

where C_2 and K_2 are as in Eqs. (17) and (18), and C_1, C_3, K_1, and K_3 are defined from $Q_1^TP(s)Q_1-R_1VF_1(s)=s^2I+sC_1+K_1$ and $Q_2^TP(s)Q_1=sC_3+K_3$. The left and right eigenvectors of A_F corresponding to each eigenvalue λ_i of interest are now needed to evaluate the condition number expression Eq. (19). Consider the right eigenvector x_{Fi} first. By inspection, this is (assuming the structural zeros and residual poles are mutually distinct) just $(\mathbf{0},x_a^H,\mathbf{0},x_b^H)^H$, where $x_i=(x_a^H,x_b^H)^H$ is the right eigenvector of A as given in Eq. (18) for the transmission zero equal to this closed-loop pole. Thus, the term $\|x\|_2$ in Eq. (19) is the same for the *zero* λ_i and the *closed-loop pole* λ_i.

The left eigenvector presents a somewhat more complicated picture. It can be seen from the block triangular structure of A_F that y_{Fi}^H is of the form $(z_a^H,y_a^H,z_b^H,y_b^H)$, where $y_i^H=(y_a^H,y_b^H)$ is the left eigenvector of A corresponding to the transmission zero λ_i, and z_a and z_b are yet to be determined. Thus, $y_{Fi}^Hx_{Fi}=y_a^Hx_a+y_b^Hx_b=y_i^Hx_i$ and $\|y_{Fi}\|_2\geq\|y_i\|_2$, so Eq. (19) shows that the condition number c_{Fi} of the closed-loop pole λ_i is related to that of the transmission zero λ_i by

$$c_{Fi}\geq c_i \tag{31}$$

The precise value of the ratio c_{Fi}/c_i depends on the quantities z_a and z_b. To determine these, consider the first and third row blocks of $y_{Fi}^HA_F=\lambda_iy_{Fi}^H$; these are $-z_a^HC_1-y_a^HC_3+z_b^H=\lambda_iz_a^H$ and $-z_a^HK_1-y_a^HK_3=\lambda_iz_b^H$, which can be rearranged to give

$$z_a^H=-y_a^H(\lambda_iC_3+K_3)[\lambda_i^2I+\lambda_iC_1+K_1]^{-1} \tag{32}$$

and

$$z_b^H=z_a^H(\lambda_iI+C_1)+y_a^HC_3 \tag{33}$$

These expressions are quite complicated in the general case but become very simple if $F_1(s)$ is chosen so as to shift all residual poles far from the transmission zero locations. As the residual poles are the eigenvalues of A_F given from $\det(s_i^2I+s_iC_1+K_1)=0$, we then have, roughly speaking, that $\det(\lambda_i^2I+\lambda_iC_1+K_1)$ must be large for all zeros λ_i. Equation (32) then implies that z_a will be small for such systems, as will z_b [Eq. (33)], so long as the open-loop structure is lightly damped. We therefore have $y_{Fi}^H\approx(\mathbf{0},y_a^H,\mathbf{0},y_b^H)$ in this case, or

$$c_{Fi}\approx c_i \tag{34}$$

Thus, if pole/zero cancellation is applied to an FSS and all residual poles are shifted far from the zero positions, the unobservable closed-loop poles will all have approximately the same condition numbers as the corresponding transmission zeros. But these were shown in the last section to have roughly the same conditioning as the open-loop poles. Combining these relations leads to the very satisfactory result that the closed-loop poles produced by pole/zero cancellation implemented in this way have approximately the same sensitivity properties as the poles of the open-loop system.

Examples

The results of the preceding sections will now be illustrated by applying them to simple flexible structures. The first system is of low enough order that the full zeros partial derivatives can be tabulated, whereas the second is more suited to a demonstration of the condition number relations of the last two sections.

Uniform Cantilever Beam

Consider a four-mode model for transverse vibration of a uniform undamped cantilever beam of length 25 m, width 0.1 m, and depth 0.01 m and constructed of aluminum (density 2.7×10^3 kg/m^3, Young's modulus 7.0×10^{10} N/m^2). If a single linear sensor/actuator pair is mounted at the free tip of the beam, the resulting poles and zeros are purely imaginary, with moduli as given in the second column of Table 1.

Of course, the transmission zeros can be seen to interlace the (well-separated) poles of this SISO structure. It is therefore expected from Eq. (8) that each zero will essentially only be sensitive to perturbations in the two poles bracketing it. This prediction is confirmed by the partial derivatives listed in Table 2; note that these do indeed all lie in the interval [0,1], as expected, and each row sum is equal to the correct value of unity. Similarly, Table 3 shows that the signs and moduli of the derivatives of the zeros with respect to the elements of the modal influence vector behave as described in the discussion

Table 1 Beam poles and zeros

Poles, rad/s	SISO zeros, rad/s	MIMO zeros, rad/s
0.0827		
0.5182	0.3637	1.0985
1.4510	1.1875	1.9089
2.8433	2.5113	

Table 2 Beam SISO zero derivatives $\partial(z_i^2)/\partial(\omega_k^2)$

Zero	$k=1$	$k=2$	$k=3$	$k=4$
$i=1$	0.5400	0.4577	0.0022	0.0001
$i=2$	0.1508	0.2279	0.6147	0.0067
$i=3$	0.0592	0.0645	0.1331	0.7432

Table 3 Beam SISO zero derivatives $\partial(z_i^2)/\partial(\phi_k^2)$

Zero	$k=1$	$k=2$	$k=3$	$k=4$
$i=1$	1.1431	−1.0524	−0.0727	−0.0180
$i=2$	3.5713	4.3898	−7.2102	−0.7509
$i=3$	6.2944	6.5672	9.4384	−22.3008

Table 4 Beam MIMO zero derivatives $\partial(z_i^2)/\partial(\omega_k^2)$

Zero	$k=1$	$k=2$	$k=3$	$k=4$
$i=1$	0.4754	0.0242	0.4764	0.0240
$i=2$	0.2168	0.1381	0.2693	0.3758

Table 5 Plate condition numbers

OL poles	SISO zeros	CL poles 1	CL poles 2	MIMO zeros
8.641	13.595	13.663	14.295	13.976
14.554	18.756	19.832	22.922	22.225
24.420	27.230	28.619	28.196	29.741
28.533	33.739	66.298	35.674	34.025
34.456	35.648	227.810	37.955	40.516
38.240	43.075	53.180	44.551	52.572
44.328	53.053	61.899	57.236	56.848
56.011	57.427	58.112	57.816	59.886
58.150	60.445	61.546	61.140	
61.715				

following Eq. (9). [The fact that each row sum is approximately zero follows from Eqs. (7) and (9) and the observation that each $|\phi_k|$ is roughly equal for this system.]

To illustrate the derivative properties of the zeros of an MIMO structure, we now add a second sensor/actuator pair halfway along the beam. The resulting two pairs of imaginary zeros are listed in the third column of Table 1 and have derivatives with respect to the natural frequencies of the structure as given in Table 4. (As previously noted, closed-form derivatives with respect to the modal matrix Φ are not now available.) Comparing Table 4 with Table 2, its SISO equivalent, reveals very simlar properties in both cases. In particular, all quantities are still positive and less than unity, and all rows still sum to 1; both these facts are as expected.

Uniform Plate

We shall now consider the uniform vertical steel plate used in Ref. 1 for pole/zero cancellation simulation studies. This plate, based on the DFVLR laboratory test article described in Ref. 21, has horizontal length 1.50 m, vertical length 2.75 m, thickness 2 mm, and isotropic material properties $E=2.0\times10^{11}$ N/m^2, $\rho=8.0\times10^3$ kg/m^3, and $\nu=0.3$. For simplicity, it is assumed to be simply supported along all four edges, leading[6] to a lowest natural frequency of 2.741 Hz and 10 modes below 20 Hz. These modes make up the model studied here; a somewhat high damping ratio of 1% is chosen for each mode to demonstrate the effects of damping on zeros sensitivity more clearly.

If a single linear sensor/actuator pair is placed at a horizontal distance 0.6 m and vertical distance 1.2 m from the lower left tip of the plate, i.e., offset slightly from the central node point, nine transmission zeros result, the lowest being at 4.321 Hz. The condition numbers [from Eq. (19)] of these poles and zeros are given as the first two columns of Table 5. It can be seen that the condition numbers of the zeros are indeed of comparable sizes to those of the open-loop poles, as expected; in fact, the approximate condition number interlacing predicted for lightly damped structures from Eq. (26)/Fig. 2 clearly applies here.

The third and fourth columns of Table 5 give the condition numbers of the unobservable closed-loop poles produced by two variants of pole/zero cancellation. As expected from Eq. (31), these condition numbers are all bounded from below by those of the corresponding transmission zeros. Column 3 corresponds to the minimum-norm feedback case, where the arbitrary $F_1(s)$ in Eq. (29) is simply taken to be zero; this minimizes the norm of the feedback gain F and so should keep the applied control forces small (but *not* minimized; see the discussion in Ref. 1). It can be seen that certain of the resulting poles have quite large condition numbers and so are very sensitive to perturbations. The discussion of the last section shows that this is a consequence of the fact that the single observable closed-loop pole is $-0.63\pm j72.93$, quite close to the zeros $-0.70\pm j71.27$ and $-0.67\pm j67.46$. If $F_1(s)$ is modified so as to shift the residual pole to $-50.00\pm j53.09$, highly damped and far from all transmission zeros, the condition numbers that result for the unobservable closed-loop poles are as given in column 4. These are clearly much closer to the condition numbers of the zeros and the open-loop poles, as predicted; furthermore, the condition number of the residual pole is reduced to 57.230 from its previous value of 244.365. This mode too is therefore considerably less sensitive to perturbations in the second case.

A second sensor/actuator pair will now be added to the structure so as to study MIMO zeros condition numbers. The position selected is midway between the lower left tip of the plate and the original sensor/actuator pair, i.e., at a horizontal coordinate of 0.3 m and a vertical coordinate of 0.6 m. The condition numbers that result are given in the final column of Table 5; it can be seen that they are again of comparable size to the condition numbers of the open-loop poles, as expected. As a final point, it can be verified that Eq. (26) could actually have been used to approximate the zeros condition numbers of this lightly damped plate (SISO and MIMO) with relative errors of only about -5×10^{-5}.

Conclusions

This paper has analyzed the sensitivity of the transmission zeros of flexible structures with compatible (physically collocated and coaxial) sensors and actuators. Two measures of sensitivity were used: first partial derivatives with respect to the modal parameters of the system and then eigenvalue condition numbers. Derivatives were shown to be useful when detailed information is required on the sensitivity of a small set of zeros, whereas condition numbers provide a more concise description of the sensitivity of the entire zeros eigenstructure.

It was proved, in both these measures, that the sensitivities of the zeros of a flexible structure are closely related to those of its poles. Furthermore, the closed-loop poles produced by applying a pole/zero cancellation control scheme to the system were shown to have condition numbers bounded from below by those of the corresponding zeros. These sensitivity measures can actually be made to approach arbitrarily close to their lower limits if all observable closed-loop poles are shifted so as to lie far from all zeros. The new zeros sensitivity results then imply that the improved performance of the closed-loop system is not achieved at the expense of increased pole sensitivity to parameter variations. These points were illustrated by application to simple single-input/single-output and multi-input/multi-output examples.

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